International Journal of Theoretical Physics, Vol. 30, No. 4, 1991

Cauchy Boundary and *b*-Incompleteness of Space-Time

Jacek Gruszczak,[1] Michael Heller,[2] and Zdzisław Pogoda[3]

Received July 9, 1990

It is shown that if a space-time (M, g) is time-orientable and its Levi-Civita connection [in the bundle of orthonormal frames over (M, g)] is reducible to an $O(3)$ structure, one can naturally select a nonvanishing timelike vector field ξ and a Riemann metric g^+ on M. The Cauchy boundary of the Riemann space (M, g^+) consists of "endpoints" of b-incomplete curves in (M, g); we call it the *Cauchy singular boundary.* We use the space-time of a cosmic string with a conic singularity to test our method. The Cauchy singular boundary of this space-time is explicitly constructed. It turns out to consist of what should be expected.

INTRODUCTION

Schmidt's (1971) b-boundary construction was once believed to provide the best definition of space-time singularities (see, Hawking and Ellis, 1973, pp. 276–284). It was a common surprise when it became manifest that the b-boundary behaves badly in some important situations. It turned out that in the closed Friedman universe both the initial and final singularities form the same, and single, point of the b-boundary (Bosshard, 1976) and that in both Friedman and Schwarzschild solutions the boundary points are not Hausdorff-separated from the corresponding space-time (Johnson, 1977). There were several attempts to cure the situation (Dodson, 1979; Clarke, 1979), but they all met other serious difficulties (see Schmidt, 1979). In practice, the b-boundary has been abandoned by relativists and often effectively replaced by the so-called causal boundary (c-boundary) (Geroch

[1]Institute of Physics, Pedagogical University, ul. Podchorążych 2, 30-084, Cracow, Poland. Cracow Group of Cosmology.
[2]Vatican Astronomical Observatory, V-00120 Vatican City State. Cracow Group of Cosmology.
[3]Institute of Mathematics, Jagiellonian University, ul. Reymonta 4, 30-059 Cracow, Poland.

et al., 1972), which, in a version improved by Penrose (1978), is able to distinguish between singular points and regular points at infinity [although some difficulties have recently been announced (Szabados, 1988, 1989) connected with "topological identifications" which are involved in the construction of c-boundaries; the construction works smoothly only for stably causal space-times]. It seems, therefore, that in such circumstances any contribution at clarifying the situation should be welcomed.

In the present work we show that by assuming that space-time (M, g) is time-orientable and that its Levi-Civita connection is reducible to an $O(3)$ structure, one can naturally select a nonvanishing timelike vector field ξ on M and a Riemann metric g^+ on M. The Cauchy boundary of the Riemann space (M, g^+) consists of "endpoints" of b-incomplete curves in (M, g); we call it the *Cauchy singular boundary.* As an example, we explicitly construct the Cauchy singular boundary of space-time due to a cosmic string producing a conic singularity (belonging to the class of the so-called quasiregular singularities).

It should be noticed that the Levi-Civita connection of a time-orientable space-time is reducible to an $O(3)$ structure if the holonomy group of this space-time is contained in (or is equal to) the $O(3)$ group. For all such space-times the singular Cauchy boundary can be uniquely constructed. The assumption that the connection is reducible to the $O(3)$ structure intervenes only in the proof of the uniqueness of our construction; it seems reasonable to believe that it could be substantially weakened.

In Section 1 we briefly discuss the methodology of the problem. In Sections 2 and 3 we show that a space-time is time-orientable if and only if the fiber bundle of orthonormal frames over it is reducible to an $O(3)$ structure and we explore some consequences of the reducibility assumption. In Section 4 the Cauchy singular boundary is defined, and in Section 5 it is constructed for the space-time of a cosmic string with the conic singularity. Some comments in Section 6 conclude our work.

1. SINGULARITIES AND SINGULAR BOUNDARIES

A Lorentz manifold (M, g) is said to be *b-complete* if every C^1-curve of a finite length (measured by a generalized affine parameter) has an endpoint in (M, g). If this is not the case, (M, g) is said to be *b-incomplete* (see e.g., Schmidt, 1971; Hawking and Ellis, 1973; Beem and Ehrlich, 1981; Dodson, 1978). The b-completeness of space-time has a good physical motivation: if a space-time is b-complete, it is both geodesically complete and complete in the sense of bounded acceleration (Beem and Ehrlich, 1981). The b-completeness property divides all Lorentz manifolds $\mathscr{L} := \{(M, g)\}$, where M is a fixed set, into two disjoint classes: $\mathscr{L}^0$ and $\mathscr{L}^s$,

$\mathscr{L} = \mathscr{L}^0 \cup \mathscr{L}^s$. Lorentz manifolds belonging to $\mathscr{L}^0$ are b-complete and are interpreted as singularity-free space-times, whereas Lorentz manifolds belonging to $\mathscr{L}^s$ are b-incomplete and are interpreted as space-time with singularities.

Let $OM(M, O(3, 1))$ be a fiber bundle of orthonormal frames over a Lorentz manifold (M, g) (in the following its connected component should be considered, if necessary), and $G^+: TOM \times TOM \to \mathbb{R}$ the so-called *Schmidt metric* on OM (Schmidt, 1971; Hawking and Ellis, 1973; Beem and Ehrlich, 1981; Dodson, 1978). The following theorem provides a mathematical foundation for the theory of b-completeness.

Theorem 1.1. A Lorentz manifold (M, g) is b-complete if and only if the Riemann space (OM, G^+) is metrically complete (m-complete, for short) (Schmidt, 1971; Hawking and Ellis, 1973; Beem and Ehrlich, 1981; Dodson, 1978).

It would be natural to organize space-time singularities into a kind of "singular boundary" ∂M of space-time (M, g), for instance, as a set of equivalence classes of b-incomplete curves in (M, g) (i.e., curves of finite length in a generalized affine parameter). Such a singular boundary should satisfy the following condition:

(*) A space-time (M, g) is singularity-free if and only if $\partial M = \emptyset$.

The concept of b-completeness does not automatically contain any construction of a singular boundary, but it should be supplemented with such a construction. It should be stressed that any construction of a singular boundary satisfying condition (*) *has equally good physical motivation.* Two such constructions were given by Schmidt (1971) and Dodson (1978). Let us briefly review the main steps of the Schmidt construction:

First, one Cauchy completes the Riemannian space (OM, G^+).

Second, one suitably extends the action of the structural group $O(3, 1)$ of the fiber bundle of orthonormal frames over M onto the Cauchy completion $\overline{OM}$ of OM.

Third, one treats $\bar{M} := \overline{OM}/O(3, 1)$ as a space-time with the singular boundary $\partial_b M := \bar{M} - M$, called *b-boundary* of space-time (M, g). Unfortunately, however, the Cauchy completion of (OM, G^+), in general, *is not a manifold* and consequently the object $\overline{OM}(\bar{M}, \pi_{\overline{OM}}, O(3, 1))$, where $\pi_{\overline{OM}}: \overline{OM} \to \bar{M}$ is a suitably extended projection $\pi_{OM}: OM \to M$, *is not a fiber bundle.* "Fibers" over b-boundary points can be strongly degenerate, and in such a case the usual theory of fiber bundles breaks down. Such a situation indeed occurs in the closed Friedman and Schwarzschild solutions (Bosshard, 1976; Johnson, 1977).

2. TIME ORIENTABILITY OF SPACE-TIME

The orientability of space-time on the local scale is one of the best empirically established facts, and globally it is a natural and philosophically appealing assumption. As is well known, a Lorentz metric exists on a paracompact manifold if and only if a continuous direction field can be defined on it. A space-time manifold is *time-orientable* if and only if this direction field can be replaced by a timelike vector field (Geroch and Horowitz, p. 225). We shall incorporate this property into our construction from the very beginning through the following result.

Proposition 2.1. A Lorentz manifold (M, g) is time-orientable if and only if the fiber bundle $OM(M, O(3,1))$ is reducible to an $O(3)$ structure $P(M, O(3))$.

Proof. Let a C^1 timelike vector field $\sigma: M \to TM$ exist on (M, g). Since any such vector field can be approximated, with any desired approximation, by a smooth timelike field (Steenrod, 1951, p. 25), we may assume that σ is smooth. By suitably normalizing it, we obtain a smooth cross section $\sigma_0: M \to UM$ of the sphere bundle UM (see Appendix B). Because of the isomorphism ψ indicated in Appendix B, the above implies the existence of a smooth cross section of the bundle $(OM \times \mathscr{H})/O(3,1)$ associated with $OM(M, O(3,1))$. Therefore, on the strength of Theorem A2 (Appendix A), there is a smooth $O(3,1)$-equivariant mapping $\phi: OM \to \mathscr{H}$ such that $\sigma_0(\pi(p)) = \langle p, \phi(p)\rangle$. Now, since $\mathscr{H}$ is a homogeneous space of the group $O(3,1)$ acting on $(\mathbb{R}^4, \eta)$, $\eta = \mathrm{diag}(-1,1,1,1)$, and $O(3)$ is the stability subgroup of the point $f_0 = (1,0,0,0)$, Theorem A1 implies that $P := \phi^{-1}(f_0)$ is an $O(3)$ structure $P(M, O(3))$.

The inverted chain of implications easily proves the converse. ■

Let (∂_i) be a local frame in M; then $(x, b^i_j\partial_i) \in OM$, where $x \in M$ and $b^i_j \in O(3,1)$. We define the so-called Sachs "projection" $\pi_{OU}: OM \to UM$ by $\pi_{OU}(x, b^i_j\partial_i) = (x, b^i_0\partial_i)$ (Sachs, 1973). The following proposition is a straightforward consequence of Theorems A1 and A2.

Proposition 2.2. There is a one-to-one correspondence between cross sections $\xi: M \to UM$ of the bundle UM and $O(3,1)$-equivariant mappings $\phi: OM \to \mathscr{H}$ of the form $\phi(p) = g^{-1}h_0$, where $h_0 \in \mathscr{H}$, $h_0 = (1,0,0,0)$, $g \in O(3,1)$, $p = p_0 g$, and p_0 is a point of an $O(3)$ structure $P(M, O(3))$. The cross section ξ assumes the form $\xi = \psi \circ \sigma$, where ψ is defined in Appendix B, and $\sigma: M \to (OM \times \mathscr{H})/O(3,1)$, $\sigma(x) = \langle p, \phi(p)\rangle$, $x = \pi_{OM}(p)$.

As another consequence of Theorems A1 and A2, we have the following results.

Proposition 2.3. If the bundle $OM(M, O(3, 1))$ is reducible to an $O(3)$ structure $P(M, O(3))$, then the mapping Π: $OM \to UM$, defined by $\Pi(p) := (\psi \circ \sigma)(\pi_{OM}(p))$, has the property

$$\Pi|_P = \pi_{OU}|_P$$

3. REDUCIBILITY OF CONNECTION

In this section we shall explore the consequences of the assumption that the connection on OM is reducible to an $O(3)$ structure.

Proposition 3.1. Let $P(M, O(3))$ be an $O(3)$ structure in $OM(M, \pi_{OM}, O(3, 1))$. If a connection Γ in OM is reducible to P, then for every b-incomplete C^1-curve $\gamma: J \to M$ there exists its horizontal lift $\tilde{\gamma}$ contained in $P(M, O(3))$, and the curve $\tilde{\gamma}$ is m-incomplete in $(P, \iota^* G^+)$, where G^+ is Schmidt's metric, and $\iota: P \to OM$ is a natural inclusion $(\iota(p) = p, p \in P)$.

Proof. This is a straightforward modification of the proof given by Dodson (1978, p. 461). ■

The following proposition is important as far as the uniqueness of our construction is concerned.

Proposition 3.2. If $P_k(M, O(3))$, $k = 1, 2$, are two $O(3)$ structures in $OM(M, O(3, 1))$ to which a connection Γ in OM is reducible, then (i) there exists $g \in O(3, 1)$ such that $P_1 = P_2 g$, and (ii) the metric spaces (P_k, D_k), $k = 1, 2$, are uniformly equivalent, where D_k are distance functions determined by the Riemann metrices $\tilde{G}_k := \iota_k^* G^+$, and $\iota_k: P_k \to OM$ are natural inclusions.

Proof. (i) Let $x \in M$ and $p_k \in \overset{-1}{\pi}_{OM}(x)$. Let further $\mathcal{H}(p_k)$ denote the holonomy bundle through p_k. There exists a $g \in O(3, 1)$ such that $\mathcal{H}(p_1) = \mathcal{H}(p_2) g$ and $p_1 = p_2 g$. It is easy to see that $P_k = \mathcal{H}(p_k) h$, $h \in O(3)$, and $P_1 = P_2 g$.

(ii) The right action of $g \in O(3, 1)$ is uniformly continuous on (OM, D), where D is a distance function determined by the Riemann metric G^+ (Dodson, 1978, p. 423). ■

4. DEFINITION OF A SINGULAR BOUNDARY

In what follows we shall assume that the considered space-time is a time-orientable Lorentz manifold with a connection reducible to an $O(3)$ structure. The reducibility of the connection uniquely distinguishes a foliation of the fiber bundle OM into $O(3)$ structures P_i, $i \in I$, to which the connection in OM is reducible. Moreover, every b-incomplete C^1-curve in M has an m-incomplete lift in P_i, for any $i \in I$ (Proposition 3.1). This means that the "endpoints" of all b-incomplete curves in M (and only they) contribute to the construction of the Cauchy boundary ∂P_i of P_i. The metric spaces (P_i, D_i) are uniformly equivalent and consequently $(\bar{P}_i, D_i)$, where $\bar{P}_i$ is the Cauchy completion of P_i, are topologically equivalent. This testifies to the fact that the construction of a singular boundary of (M, g) should be connected with any of the $O(3)$ structures P_i. Such a construction will take into account all b-incomplete curves in M, and will not depend on the choice of a particular $O(3)$ structure P_i.

Let all symbols be as above, and let us consider two metric spaces (OM, D) and $(P, \tilde{D})$, where D and $\tilde{D}$ are distance functions determined by metrics G^+ and $\tilde{G}^+ := \iota^* G^+$ on OM and P, respectively.

Proposition 4.1. If $\{x_n\}$ is a Cauchy sequence in $(P, \tilde{D})$, then the sequence $\{y_n\}$ in UM, defined by $y_n := \Pi(x_n)$, is a Cauchy sequence in (UM, d_s), where d_s is the distance function determined by the Sachs metric g_s^+ in UM (see Appendix B).

Proof. The theorem is a direct consequence of the fact that $\Pi|_P = \pi_{UM}|_P$ and that the "projection" π_{UM} is uniformly continuous (Sachs, 1973). ■

Corollary 4.2. If the bundle $OM(M, O(3, 1))$ is reducible to an $O(3)$ structure $P(M, O(3))$, then:

(i) On M there exists a distinguished uniform structure defined by the distance function $d: M \times M \to \mathbb{R}^+ \cup \{0\}$ which is determined by the Riemann metric $g^+ := \xi^* g_s^+$, where $\xi: M \to UM$ is a cross section as given by Proposition 2.2.

(ii) The mapping f of metric spaces $(P, \tilde{D})$ onto (M, d), $f: P \to M$, defined by $f = \pi_{UM} \circ \Pi|_P$, is uniformly continuous.

Let (M, g) be a time-orientable Lorentz manifold such that a connection Γ in $OM(M, O(3, 1))$ is reducible to an $O(3)$ structure $P(M, O(3))$.

Definition 4.3. The Cauchy boundary $\partial M := \bar{M} - M$ of the metric space (M, d) will be called the *Cauchy singular boundary* of (M, g).

Proposition 4.4. The Cauchy singular boundary ∂M of (M, g) satisfies condition $(*)$.

Proof. If (M, g) is b-complete, (OM, G^+) is m-complete (Theorem 1.1). If (M, G^+) is m-complete, $(P, \iota^* G^+)$ is m-complete, and Corollary 4.2 implies that (M, d) is m-complete. Therefore $\partial M = \emptyset$. ■

5. TEST OF THE SINGULAR CAUCHY BOUNDARY CONSTRUCTION

To test our method, we shall construct the singular Cauchy boundary of space-times with metrics representing the external gravitational field of a cosmic string. Topological properties of these space-times are well known [since the singularities are introduced by construction (see Staruszkiewicz, 1983; Ellis and Schmidt, 1977)] and we shall see that the singular Cauchy boundary of these space-times exactly reproduces the known structures.

Let us consider a Lorentz manifold (M, g) with the metric

$$g = -dt^2 + d\rho^2 + \rho^2\, d\phi^2 + dz^2 \tag{5.1}$$

where t, $z \in \mathbb{R}$, $\rho \in (0, \infty)$, $\phi \in (0, 2\pi - \Delta)$, $\Delta \in (0, 2\pi)$, and M is a domain of $\mathbb{R}^4$ which originates by removing from it a wedge $A = \mathbb{R}^2 \times K$, K being an angle in the plane (x, y), $x = \rho \cos\phi$, $y = \rho \sin\phi$; Δ is a measure of K. We can see that (M, g) is the Minkowski space-time from which the wedge A has been removed. Of course, ∂M represents a regular boundary (Ellis and Schmidt, 1977).

Let us choose the vector field $\xi = (1, 0, 0, 0)$ in (M, g). It can be easily checked that in this case the connection is reducible to the $O(3)$ structure P corresponding to ξ and that

$$g^+ := \xi^* g_s^+ = dt^2 + d\rho^2 + \rho^2\, d\phi^2 + dz^2$$

The direct computation shows that the singular Cauchy boundary in this case is indeed equal to ∂M.

Let now (M_1, g_1) be the space-time corresponding to the external gravitational field of a cosmic string

$$g_1 = -dt^2 + a^2\, dr^2 + r^2\, d\phi^2 + dz^2$$

where t, $z \in \mathbb{R}$, $\phi \in (0, 2\pi)$, $r \in (0, \infty)$, $a \in \mathbb{R}$, and $M_1 = \mathbb{R}^2 \times C_0$, C_0 being a connected component of a cone without the vertex S_0. M_1 originates from M (see above) by a suitable gluing up of the edges of the wedge A. (M_1, g_1) has the *quasiregular singularity* $S_0 \times \mathbb{R}^2$.

Similarly as above, we put $\xi = (1, 0, 0, 0)$. The connection is again reducible to the $O(3)$ structure P corresponding to ξ. Moreover,

$$g_1^+ = \xi^* g_{1s}^+ = +dt^2 + a^2\, dr^2 + r^2\, d\phi^2 + dz^2$$

It can be easily checked that the Riemannian manifold has the Cauchy (singular) boundary $S_0 \times \mathbb{R}^2$.

These examples show the consistency of our construction of the Cauchy singular boundary. The construction can be also applied to space-times with curvature singularities. It excludes from the very beginning a non-Hausdorff behavior of boundary points. We shall deal with these problems in a subsequent work.

6. SPACE-TIME AS A DYNAMICAL SYSTEM

For a time-orientable Lorentz manifold (M, g), there is no uniquely determined Cauchy completion. This is a consequence of the fact that M is a metrizable but not metric space. In other words, (M, g) has no uniquely defined uniform structure and consequently it admits many nonequivalent Cauchy completions. This problem has been discussed in Gruszczak (1990), where it has been also shown that if one selects a uniform structure for (M, g) [or equivalently if one chooses a particular Cauchy boundary of (M, g) from among many admissible ones], possibly by appealing to physical arguments, one can define space-time in such a way that its topology, uniform structure, and time-orientability will simultaneously be determined. In the present work we have shown that the assumptions of time-orientability and reducibility of connection to an $O(3)$ structure uniquely select a Cauchy completion and that the corresponding Cauchy boundary is physically well motivated [condition $(*)$ is satisfied].

As above, let (M, g) be a space-time. According to Propositions 2.2 and 3.2, the connection Γ in OM distinguishes the class of vector fields $\Xi := \{\xi_i: M \to UM,\ i \in I\}$. Simultaneously, on the strength of Corollary 4.2(i), there exists the uniquely distinguished Riemann metric $g_i^+: TM \times TM \to \mathbb{R}$ given by $g_i^+ = \xi^{i*} g_s^+$. Let us choose $i_0 = I$. One can easily notice that the triple $(M, g_{i_0}^+, \xi_{i_0})$ is a metric dynamical system. The results of the above analysis well motivate the following.

Definition 6.1. A metric dynamical system $(M, g_{i_0}^+, \xi_{i_0})$ is said to be a *space-time with the Cauchy boundary* (*C-space-time*, for short). This boundary is uniquely determined as a Cauchy boundary of the Riemann manifold $(M, g_{i_0}^+)$ [for details see Gruszczak (1990)].

It is important to notice that in the definition of C-space-time all properties of space-time as the pair (M, g) are preserved because of the fact that g is uniquely determined by the relationship $g_i^+ = \xi^{i*} g_s^+$. This procedure (and the metric uniform structure determined by it) does not depend of the choice of $i \in I$. Of course, the Cauchy boundary of the Riemann manifold $(M, g_{i_0}^+)$, which defines the boundary of the C-space-time $(M, g_{i_0}^+, \xi_{i_0})$, is a Cauchy singular boundary ∂M of the space-time (M, g) in the sense of Definition 4.3.

Let us notice that the same assumptions naturally select a timelike vector field ξ on M, and a Riemann metric g^+ on M, and consequently, for the considered class of space-times, the dynamical system (M, g^+, ξ) is equivalent to the pair (M, g). In many physical applications (including quantization of gravity) it might be much easier to deal with the Riemann manifold equipped with a nonvanishing timelike vector field (M, g^+, ξ) than with the Lorentz manifold (M, g).

APPENDIX A. REDUCIBLE BUNDLES AND CROSS SECTIONS

Let $Q(M, G)$ be a principal fiber bundle.

Definition A1. Every principal fiber bundle $P(M, H)$ such that (i) P is a submanifold of Q, (ii) H is a subgroup of G, and (iii) the mapping $I: P(M, H) \to Q(M, G)$, $I = (\iota_P, id_M, \iota_G)$, where $\iota_P : P \to Q$, $\iota_G: H \to G$ are natural inclusions, is a homeomorphism of principal fiber bundles, is called the *H structure* in $Q(M, G)$. $Q(M, G)$ is said to be *reducible* to the H structure $P(M, H)$.

Let K and L be manifolds, and G a group acting on K to the right and on L to the left.

Definition A2. A smooth mapping $f: K \to L$ is called *G-equivariant* if, for any $k \in K$ and any $g \in G$, one has $f(kg) = g^{-1}f(k)$.

Theorem A1. Let $Q(M, G)$ be a principal fiber bundle, H a closed subgroup of G, and F a homogeneous space (for G) such that, for any $f \in F$, the isotropy group of f is $H_f = H$. There is a one-to-one correspondence between *H structures* $P(M, H)$ in $Q(M, G)$ and G-equivariant mappings $\phi : Q \to F$. Moreover:

(1) If $P(M, H)$ is a H structure in $Q(M, G)$, then the corresponding equivariant mapping $\phi : Q \to F$ is given by $\phi(p) = g^{-1}f$, where p is of the form $p = p_0 g$, for any $p_0 \in P$.

(2) If $\phi : Q \to F$ is a G-equivariant mapping, then $\phi^{-1}(f)$ is the corresponding H structure in $Q(M, G)$.

Theorem A2. There is a one-to-one correspondence between smooth cross sections of a fiber bundle $E(M, G, F)$, with the standard fiber F, associated with the principal fiber bundle $Q(M, G)$ and G-equivariant mappings $\phi : Q \to F$. If a cross section $\sigma : M \to E$ corresponds to the mapping $\phi : Q \to F$, then

$$\sigma(\pi(p)) = \langle p, \phi(p) \rangle \in E$$

(See, for instance, Crittenden, 1962; Kobayashi and Nomizu, 1963; Gancarzewicz, 1987.)

APPENDIX B. SPHERE BUNDLE

Let the sphere bundle UM be defined by

$$UM = \{(x, X) \in TM: X \text{ is unit and timelike}\}$$

In the original work by Sachs (1973) it is additionally assumed that X is future-directed; we do not need this assumption. The bundle UM is isomorphic (in the bundle sense) with the bundle $(OM \times \mathscr{H})/O(3, 1)$ associated with the fiber bundle of pseudoorthonormal frames $OM(M, O(3, 1))$, where $\mathscr{H} = \{h \in \mathbb{R}^4: \eta_{ik}h^ih^k = -1\}$, $\eta = \text{diag}(-1, 1, 1, 1)$. The isomorphism is given by

$$\psi: \quad (OM \times \mathscr{H})/O(3, 1) \to UM$$

$$\psi(\langle p, h \rangle) = v_i h^i$$

$$p = (v_0, \ldots, v_3) \in OM, \qquad h = (h^0, \ldots, h^3) \in \mathscr{H}$$

Let $\pi_U: UM \to M$ be a projection defined by the inclusion map $\iota: UM \hookrightarrow TM$. On UM there exists a unique Riemann metric g_s defined in the following way:

$$g_s(Y, Y) = g(d\pi_U Y, d\pi_U Y) + 2[g(d\pi_U Y, \iota(X))]^2 + g(\Gamma_{d\pi_U Y}\iota, \Gamma_{d\pi_U A}\iota)$$

where g and Γ are the Lorentz metric and Levi-Civita connection on M, correspondingly; g_s will be called the *Sachs metric.* (See Sachs, 1973; Dodson, 1978.)

ACKNOWLEDGMENTS

The work of J.G. and Z.P. was supported by the Polish Interdisciplinary Research Program CPBP 01.03.

REFERENCES

Beem, J., and Ehrlich, P. (1981). *Global Lorentzian Geometry,* Dekker.

Bosshard, B. (1976). *Communications in Mathematical Physics,* **46**, 263.

Clarke, C. J. S. (1979). *General Relativity and Gravitation,* **10**, 977.

Crittenden, R. (1962). *Quarterly Journal of Mathematics* (Oxford), **13**, 285.

Dodson, C. T. J. (1978). *International Journal of Theoretical Physics,* **17**, 389.

Dodson, C. T. J. (1979). *General Relativity and Gravitation,* **10**, 969.

Ellis, G. F. R., and Schmidt, B. (1977). *General Relativity and Gravitation,* **8**, 915.

Gancarzewicz, J. (1987). *Differential Geometry,* Polish Scientific Publishers [in Polish].

Geroch, R., and Horowitz, G. T. (1979). In *General Relativity—An Einstein Centenary Survey,* S. W. Hawking and W. Israel, eds., Cambridge University Press, Cambridge, p. 212.

Geroch, R., Kronheimer, E. H., and Penrose, R. (1972). *Proceedings of the Royal Society of London A,* **327**, 545.

Gruszczak, J. (1990). *International Journal of Theoretical Physics,* **29**(1), (1990).

Hawking, S. W., and Ellis, G. F. R. (1973). *Large Scale Structure of Space-Time,* Cambridge University Press, Cambridge.

Johnson, R. A. (1977). *Journal of Mathematical Physics,* **18**, 898.

Kobayashi, S., and Nomizu, K. (1963). *Foundations of Differential Geometry,* Vol. 1, Interscience, New York.

Penrose, R. (1978). *Theoretical Principles in Astrophysics and Relativity*, N. R. Lebovitz, W. H. Rein, and P. O. Vandervoort, eds., University of Chicago Press, Chicago, Illinois.

Sachs, R. K. (1973). *Communications in Mathematical Physics*, **33**, 215.

Schmidt, B. G. (1971). *General Relativity and Gravitation*, **1**, 269.

Schmidt, B. G. (1979). *General Relativity and Gravitation*, **10**, 981.

Staruszkiewicz, A. (1963). *Acta Physica Polonica*, **24**, 734.

Steenrod, N. (1951). *Topology of Fiber Bundles*, Princeton University Press, Princeton, New Jersey.

Szabados, L. B. (1988). *Classical and Quantum Gravity*, **5**, 121.

Szabados, L. B. (1989). *Classical and Quantum Gravity*, **6**, 77.